A GENERALIZATION OF THE TAYLOR COMPLEX CONSTRUCTION

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Abstract. Given multigraded free resolutions of two monomial ideals we construct a multigraded free resolution of the sum of the two ideals.

Introduction

Let K be a field, $S = K[x_1, \dots, x_n]$ the polynomial ring in n variables over K, and let I and J be two monomial ideals in S. Suppose that $\mathbb{F}$ is a multigraded free S-resolution of S/I and $\mathbb{G}$ a multigraded free S-resolution of S/J. In this note we construct a multigraded free resolution of $S/(I+J)$ which we denote by $\mathbb{F} * \mathbb{G}$. It follows from our construction that $\beta_i(S/(I+J)) \leq \sum_{j=0}^{i} \beta_j(S/I)\beta_{i-j}(S/J)$ for all $i \geq 0$. Here $\beta_i(M)$ denotes the ith Betti number of a graded S-module M, that is, the K-dimension of $\mathrm{Tor}_i^S(K, M)$.

The inequality for the Betti-numbers implies in particular that $\operatorname{proj\,dim}(I+J) \leq \operatorname{proj\,dim}(I) + \operatorname{proj\,dim}(J) + 1$. The numerical data of the complex $\mathbb{F} * \mathbb{G}$ also yield the inequality $\operatorname{reg}(I+J) \leq \operatorname{reg}(I) + \operatorname{reg}(J) - 1$. Similar inequalities hold for the projective dimension and the regularity of $I \cap J$, see Section 3. The inequality for the regularity has first been conjectured by Terai [7]. He also proved this inequality in a special case.

In the squarefree case these inequalities have first been proved by Kalai and Meshulam [5]. The construction of the complex $\mathbb{F} * \mathbb{G}$ was inspired by the work of Kalai and Meshulam. In fact, the first author informed me that the above mentioned inequalities for the projective dimension and the regularity of sums and intersections of squarefree monomial ideals follow from certain inequalities proved in [5] concerning the d-Leray properties of the union and intersection of simplicial complexes. Thus our construction provides an algebraic explanation of these inequalities.

One should note that for example the inequality $\operatorname{proj\,dim}(I+J) \leq \operatorname{proj\,dim}(I) + \operatorname{proj\,dim}(J) + 1$, as well as all the other inequalities, are wrong for arbitrary graded ideals.

We would also like to mention that the Taylor resolution (cf.[4]) is a special case of our construction. The Taylor resolution is a multigraded free resolution for monomial ideals. It has a uniform structure, but in most cases, the Taylor resolution is non-minimal. In the frame of our construction the Taylor resolution can be described as follows: if $I \subset S$ is a monomial ideal with the minimal set of monomial generators

$\{u_1,\ldots,u_r\}$, and $\mathbb{F}_j$ is the graded minimal free resolution of the principal ideal (u_j) for $j=1,\ldots,r$, then $\mathbb{F}_1 * \mathbb{F}_2 * \cdots * \mathbb{F}_r$ is the Taylor resolution of S/I.

1. The construction

Let K be field, $S=K[x_1,\ldots,x_n]$ a polynomial ring and $I\subset S$ a monomial ideal. Then S/I admits a multigraded minimal free S-resolution

$$\mathbb{F}\colon 0 \longrightarrow F_p \xrightarrow{\varphi_p} F_{p-1} \xrightarrow{\varphi_{p-1}} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow 0,$$

that is, one has

(i) $H_0(\mathbb{F})=S/I$;

(ii) $F_i=\bigoplus_j S(-a_{ij})$ with $a_{ij}\in\mathbb{Z}^n$ for all i;

(iii) the differentials φ_i are homomorphisms of multigraded modules.

We define a partial order on $\mathbb{Z}^n$ by saying that $b\leq a$ for $a,b\in\mathbb{Z}^n$, if b is componentwise less than a.

For all i let B_i be a multihomogeneous basis of F_i. Then $F_i=\bigoplus_{g\in B_i} Sg$, and the differential $\varphi_i\colon F_i\to F_{i-1}$ can be described by the equations

$$\varphi_i(g)=\sum_{h\in B_{i-1}} a_{gh}u_{gh}h, \tag{1}$$

where $a_{gh}\in K$ with $a_{gh}=0$ if $\deg g<\deg h$, and where the coefficient u_{gh} is the unique monomial in S with $\deg g=\deg u_{gh}+\deg h$ whenever $\deg g\geq \deg h$.

For a homogeneous element f in a multigraded module we denote by u_f the unique monomial with $\deg u_f=\deg f$. Then for all i, all $g\in B_i$ and all $h\in B_{i-1}$ with $a_{gh}\neq 0$ we have $u_{gh}=u_g/u_h$.

Now let $J\subset S$ be another monomial ideal with minimal multigraded free resolution $\mathbb{G}$ whose differential ψ is given by

$$\psi_i(e)=\sum_{f\in B'_{i-1}} b_{ef}u_{ef}f \quad \text{for} \quad e\in B'_i, \tag{2}$$

where for each i, B'_i is a homogeneous basis of G_i.

We are going to construct an acyclic multigraded complex $\mathbb{F}*\mathbb{G}$ of free S-modules which provides a free resolution of $S/(I+J)$.

Let F be a multigraded free S-module with homogeneous basis B and G a multigraded free S-module with homogeneous basis B'. We let $F*G$ be the multigraded free S-module with a basis given by the symbols $f*g$ where $f\in B$ and $g\in B'$. The multidegree of $f*g$ is defined to be

$$\deg f*g=[u_f,u_g],$$

where for two monomials $u,v\in S$ we denote by $[u,v]$ the least common multiple of u and v. Denoting by (u,v) the greatest common divisor of u and v, the map

$$j\colon F\otimes G\to F*G,\quad f\otimes g\mapsto (u_f,u_g)f*g \tag{3}$$

is a multigraded monomorphism.

Now we are ready to define $\mathbb{F} * \mathbb{G}$: we let

$$(\mathbb{F} * \mathbb{G})_i = \bigoplus_{\substack{j,k \\ j+k=i}} F_j * G_k,$$

and define the differential

$$\alpha_i \colon (\mathbb{F} * \mathbb{G})_i \longrightarrow (\mathbb{F} * \mathbb{G})_{i-1}$$

by the equation

$$\alpha_i(g * e) = \sum_{h \in B_{j-1}} a_{gh} u_{ghe} h * e + (-1)^j \sum_{f \in B'_{k-1}} b_{ef} u_{gef} g * f$$

where $g \in B_j$ and $e \in B'_k$ with $j + k = i$. Here

$$u_{ghe} = [u_g, u_e]/[u_h, u_e] \quad \text{and} \quad u_{gef} = [u_g, u_e]/[u_g, u_f].$$

Extending the multigraded monomorphism (3) naturally to $\mathbb{F} \otimes \mathbb{G}$ we obtain a monomorphism of multigraded modules

$$(4) \qquad j \colon \mathbb{F} \otimes \mathbb{G} \longrightarrow \mathbb{F} * \mathbb{G}$$

with the property that $\alpha \circ j = j \circ \partial$, where ∂ denotes the differential of $\mathbb{F} \otimes \mathbb{G}$. Since j becomes an isomorphism after localization with respect to all variables, it follows that $\alpha \circ \alpha = 0$, so that $\mathbb{F} * \mathbb{G}$ is a complex of multigraded S-modules.

2. Acyclicity

The aim of this section is to prove the following

Theorem 2.1. *Let I and J be monomial ideals in $S = K[x_1, \ldots, x_n]$ with multigraded free S-resolutions $\mathbb{F}$ and $\mathbb{G}$, respectively. Then $\mathbb{F} * \mathbb{G}$ is a multigraded free S-resolution of $S/(I + J)$.*

For the proof of this theorem we need to consider polarization of monomial ideals. Let $I = (u_1, \ldots, u_m)$ with $u_j = x_1^{a_{j1}} \cdots x_n^{a_{jn}}$, and let a be the maximum of the exponents a_{ij}. We denote by $\sup u_j$ the set of elements $i \in [n]$ with $a_{ji} \neq 0$. Consider the polynomial ring T over S in the variables y_{ik}, $i = 1, \ldots, n$ and $k = 1, \ldots, r$ with $r \geq a$. The *polarization of* I is the squarefree monomial ideal $I^{\wp} \subset T$ whose generators are the monomials

$$u_j^{\wp} = \prod_{i \in \sup u_j} y_{i1} y_{i2} \cdots y_{i,a_{ji}}, \quad j = 1, \ldots, m.$$

It is known that

$$(5) \quad \mathbf{z} = x_1 - y_{11}, \ldots, x_1 - y_{1r}, x_2 - y_{21}, \ldots, x_2 - y_{2r}, \ldots, x_n - y_{n1}, \ldots x_n - y_{nr}$$

is a regular sequence on $T/I^{\wp}$ with $(T/I^{\wp})/(\mathbf{z})(T/I^{\wp}) \cong S/I$.

Let $\mathbb{F}$ be a minimal multigraded free resolution of S/I. We shall need the following result of Sbarra [6], whose proof we indicate for the convenience of the reader.

Proposition 2.2. *Let $\mathbb{F}^\wp$ be a minimal multigraded free resolution of $T/I^\wp$ and for each i, let $B_i^\wp$ be a multihomogeneous basis of $\mathbb{F}_i^\wp$. Then for each i, there exists a multihomogeneous basis B_i of F_i and a bijection $B_i^\wp \to B_i$, $f \mapsto \bar{f}$ with the property that $u_f = u_{\bar{f}}^\wp$ for all $f \in B_i^\wp$. In other words, the shifts in the resolution of $\mathbb{F}^\wp$ are obtained from the shifts in $\mathbb{F}$ by polarization.*

Proof. Notice that $\mathbb{F}^\wp/(\mathbf{z})\mathbb{F}^\wp$ is a minimal graded free S-resolution of S/I since the sequence $\mathbf{z}$ (see (5)) is regular on $T/I^\wp$. With respect to the coarse multigrading on T which assigns to each y_{ik} and to each x_i the multidegree ε_i where ε_i is ith vector of the canonical basis of $\mathbb{Q}^n$, the sequence $\mathbf{z}$ is even homogeneous, so that $\mathbb{F}^\wp/(\mathbf{z})\mathbb{F}^\wp$ is a multigraded complex of S-modules, and hence as a multigraded complex is isomorphic to $\mathbb{F}$. Thus we may identify $\mathbb{F}^\wp/(\mathbf{z})\mathbb{F}^\wp$ with $\mathbb{F}$.

Let $f \in B_i^\wp$. We denote the residue class of f in $\mathbb{F}^\wp/(\mathbf{z})\mathbb{F}^\wp$ by $\bar{f}$, and set $B_i = \{\bar{f} \colon f \in B_i^\wp\}$. Then for all $i \geq 0$, B_i is a multihomogeneous basis of F_i.

Since $I^\wp$ is a squarefree monomial ideal, each u_f is a squarefee monomial. In other words, $u_f = \prod_{i=1}^n \prod_{j \in A_i} y_{ij}$ with certain $A_i \subset \{1, \ldots, r\}$. It follows that $u_{\bar{f}} = \prod_{i=1}^n x_i^{|A_i|}$. If we can show that each A_i is of the form $A_i = \{1, \ldots, k_i\}$ for some k_i, then $u_{\bar{f}}^\wp = u_f$, as desired.

Since the Taylor complex $\mathbb{T}$ of $I^\wp$ is a multigraded free T-resolution of $T/I^\wp$, while $\mathbb{F}^\wp$ is a *minimal* multigraded free T-resolution of $T/I^\wp$, we conclude that $\mathbb{F}^\wp$ is isomorphic to a multigraded direct summand of $\mathbb{T}$. Let $G(I^\wp)$ be the unique minimal monomial set of generators of $I^\wp$. The shifts of $\mathbb{T}$ are the least common multiples of subsets of $G(I^\wp)$. Since each of the generators of $I^\wp$ is of the form $\prod_{i=1}^n \prod_{j=1}^{k_i} y_{ij}$, it follows that all shifts of $\mathbb{T}$, and hence all shifts of $\mathbb{F}^\wp$ are of the same form, as desired. $\square$

Proof of Theorem 2.1. Obviously one has $H_0(\mathbb{F} * \mathbb{G}) = S/(I + J)$. In order to show that $\mathbb{F} * \mathbb{G}$ is acyclic, we first treat the case that I and J are squarefree monomial ideals and that the resolutions $\mathbb{F}$ and $\mathbb{G}$ are minimal. We consider the following complex filtration of $\mathbb{F} * \mathbb{G}$:

$$0 = \mathcal{F}^0(\mathbb{F} * \mathbb{G}) \subset \mathcal{F}^1(\mathbb{F} * \mathbb{G}) \subset \cdots \subset \mathcal{F}^n(\mathbb{F} * \mathbb{G}) = \mathbb{F} * \mathbb{G},$$

where

$$\mathcal{F}^j(\mathbb{F} * \mathbb{G}) = \bigoplus_{i \leq j} \mathbb{F} * G_i.$$

The factor complexes $\mathcal{F}^j(\mathbb{F} * \mathbb{G})/\mathcal{F}^{j-1}(\mathbb{F} * \mathbb{G})$ are isomorphic to $\mathbb{F} * G_j$ with differential given by

$$F_i * G_j \longrightarrow F_{i-1} * G_j, \quad g * e \mapsto \sum_{h \in B_{i-1}} a_{gh} u_{ghe} h * e \tag{6}$$

Here we use the assumptions and notation introduced in the previous section.

The E^2-terms of the first quadrant spectral sequence induced by the filtration $\mathcal{F}$ are given by the homology of the factor complexes, that is,

$$E_{i,j}^2 = H_j(\mathbb{F} * G_i) \quad \text{for all} \quad i, j.$$

We claim that each of these factor complexes is acyclic. To this end we first notice that $\mathbb{F} * G_j$ is the direct sum of the complexes $\mathbb{F} * Se$ with $e \in B'_j$. In other words,

$$\mathbb{F} * G_j \cong \bigoplus_{e \in B'_j} \mathbb{F} * Se.$$

Thus it suffices to show that each of the complexes $\mathbb{F} * Se$ is acyclic.

The complex homomorphism (4) restricts to the complex homomorphism

$$j\colon \mathbb{F} \longrightarrow \mathbb{F} * Se, \quad g \mapsto (u_g, u_e) g * e \quad \text{for} \quad g \in \bigcup_i B_i. \tag{7}$$

Thus after localization we have an isomorphism of complexes

$$(\mathbb{F})_{u_e} \cong (\mathbb{F} * Se)_{u_e}.$$

In particular, it follows that $(\mathbb{F} * Se)_{u_e}$ is acyclic.

Without loss of generality we may assume that $u_e = \prod_{i=k+1}^n x_i$. Now since the differentials of $\mathbb{F} * Se$ are given by

$$F_i * Se \to F_{i-1} * Se, \quad g * e \mapsto \sum_{h \in B_{i-1}} a_{gh} u_{ghe} h * e \quad \text{with} \quad u_{ghe} = [u_g, u_e]/[u_h, u_e],$$

we see that all the monomial entries u_{ghe} of the differentials are monomials in $S' = K[x_1, \dots, x_k]$, so that $\mathbb{F} * Se \cong \mathbb{H} \otimes_{S'} S$ where $\mathbb{H}$ is a multigraded complex of free S'-modules. (Here is where we use that I and J are squarefree). Hence

$$(\mathbb{F} * Se)_{u_e} \cong (\mathbb{H} \otimes_{S'} S) \otimes_S S_{u_e} \cong \mathbb{H} \otimes_{S'} S'[x_{k+1}^{\pm 1}, \dots, x_n^{\pm 1}].$$

Since $(\mathbb{F} * Se)_{u_e}$ is acyclic and the since the extension $S' \subset S'[x_{k+1}^{\pm 1}, \dots, x_n^{\pm 1}]$ is faithfully flat, it follows that $\mathbb{H}$ is acyclic. Then, using the fact that the extension $S' \subset S$ is flat, we conclude that $\mathbb{F} * Sg \cong \mathbb{H} \otimes_{S'} S$ is acyclic, as desired.

Note that $H_0(\mathbb{F} * Sg) = (S/I_g)g$, where I_g is generated by the monomials $[u, u_g]/u_g$ with $u \in G(I)$. Here we denote as usual by $G(L)$ the unique minimal set of monomial generators of a monomial ideal L. Thus our calculations have shown that

$$E^2_{i,j} = \begin{cases} 0, & \text{if} \quad j > 0, \\ \bigoplus_{g \in B'_i} (S/I_g)g, & \text{if} \quad j = 0. \end{cases}$$

Therefore $\mathbb{F} * \mathbb{G}$ will be acyclic if the complex

$$\tilde{\mathbb{G}}\colon 0 \longrightarrow \bigoplus_{g \in B'_q} (S/I_g)g \longrightarrow \cdots \longrightarrow \bigoplus_{g \in B'_2} (S/I_g)g \longrightarrow \bigoplus_{g \in B'_1} (S/I_g)g \longrightarrow S/I \longrightarrow 0,$$

is acyclic, where the differentials $\tilde{\psi}_i$ of $\tilde{\mathbb{G}}$ are induced by those of $\mathbb{G}$.

In order to prove the acyclicity of $\tilde{\mathbb{G}}$, let $i > 0$ and $z \in \tilde{G}_i$ be a multihomogeneous element with $\tilde{\psi}_i(z) = 0$, and let $w \in G_i$ be a multihomogeneous element with $\varepsilon(w) = z$, where $\varepsilon\colon G_i \to \tilde{G}_i$ is the canonical epimorphism. Then

$$\psi_i(w) \in \bigoplus_{h \in B'_{i-1}} I_h h,$$

and we have to show that $w \in \operatorname{Im}(\psi_{i+1}) + \bigoplus_{g \in B'_i} I_g g$.

We have $w = \sum_g c_g w_g g$ with $c_g \in K$ and w_g a monomial with $\deg w_g + \deg g = \deg w$ for all g with $c_g \neq 0$. Then

$$\psi_i(w) = \sum_g c_g w_g (\sum_h b_{gh} u_{gh} h) = \sum_h (\sum_g c_g b_{gh} w_g u_{gh}) h.$$

The monomial $w_g u_{gh}$ only depends on w and on h (and not on g). We therefore denote it by v_h, and obtain

$$\psi_i(w) = \sum_h (\sum_g c_g b_{gh}) v_h h.$$

If $\psi_i(w) = 0$, then $w \in \mathrm{Im}(\psi_{i+1})$ since $\mathbb{G}$ is acyclic. Otherwise, there exists $h \in B'_{i-1}$ such that $\sum_g c_g b_{gh} \neq 0$. For this h one has $v_h \in I_h$, and there exists $e \in B'_i$ with $c_e b_{eh} \neq 0$. Since $v_h = w_e u_{eh} = w_e(u_e/u_h)$ it follows that $w_e u_e \in I_h u_h \subset I$, so that $w_e \in I_e$. Since on the other hand $\psi_i(z_e e) \in \bigoplus_{h \in B'_{i-1}} I_h h$ for any monomial $z_e \in I_e$, it follows for $w' = \sum_{g,\, g \neq e} c_g w_g g$ that

$$\psi_i(w') = \psi(w) - \psi(c_e w_e e) \in \bigoplus_{h \in B'_{i-1}} I_h h.$$

Hence using induction on the number of summands in w we may assume that $w' \in \mathrm{Im}(\psi_{i+1}) + \bigoplus_{g \in B'_i} I_g g$, which yields the desired conclusion since $c_e w_e e \in \bigoplus_{g \in B'_i} I_g g$.

Next we consider the case that I and J are arbitrary monomial ideals in S and that the resolutions $\mathbb{F}$ and $\mathbb{G}$ are minimal. We use polarization, to reduce this more general case to the case of squarefree monomial ideals.

Assume that the differential of $\mathbb{F}^\wp$ is given by

$$\varphi_i(g) = \sum_{h \in B^\wp_{i-1}} a_{gh} u_{gh} h,$$

and that of $\mathbb{G}^\wp$ is given by

$$\psi_i(e) = \sum_{f \in B'^\wp_{i-1}} b_{ef} u_{ef} f.$$

with multihomogeneous bases $B^\wp_i$ and $B'^\wp_i$.

Then the differential of $\mathbb{F}^\wp * \mathbb{G}^\wp$ is given by

$$\alpha_i(g * e) = \sum_{h \in B^\wp_{i-1}} a_{gh} u_{ghe} h * e + (-1)^j \sum_{f \in B'^\wp_{i-1}} b_{ef} u_{gef} g * f$$

for $g \in B^\wp_j$ and $f \in B'^\wp_k$, where

$$u_{ghe} = [u_g, u_e]/[u_h, u_e] \quad \text{and} \quad u_{gef} = [u_g, u_e]/[u_g, u_f].$$

It follows that the differential of $(\mathbb{F}^\wp * \mathbb{G}^\wp)/(\mathbf{z})(\mathbb{F}^\wp * \mathbb{G}^\wp)$ is given by

$$\overline{\alpha}_i(\overline{g * e}) = \sum_{h \in B^\wp_{i-1}} a_{gh} \overline{u}_{ghe} \overline{h * e} + (-1)^j \sum_{f \in B'^\wp_{i-1}} b_{ef} \overline{u}_{gef} \overline{g * f}.$$

Here $^{-}$ denotes the residue class of an element modulo $\mathbf{z}$. Now observe that

$$\overline{u}_{ghe} = \overline{[u_g, u_e]/[u_h, u_e]} = \overline{[u_g, u_e]}/\overline{[u_h, u_e]} = [\overline{u}_g, \overline{u}_e]/[\overline{u}_h, \overline{u}_e] = [u_{\bar{g}}, u_{\bar{e}}]/[u_{\bar{h}}, u_{\bar{e}}] = u_{\bar{g}\bar{h}\bar{e}},$$

and similarly, $\overline{u}_{gef} = u_{\bar{g}\bar{e}\bar{f}}$. For the third equation we used that $\overline{[u, v]} = [\bar{u}, \bar{v}]$ for monomials u and v which are of the form $\prod_{j=1}^{n}\prod_{i=1}^{k_i} y_{ji}$. But this condition is satisfied in our case since by Proposition 2.2 the monomials u_g, u_h and u_e are the polarizations of the monomials $u_{\bar{g}}$, $u_{\bar{h}}$ and $u_{\bar{e}}$, respectively. For the fourth equation we used, also shown in Proposition 2.2, that $\overline{u}_f = \overline{u_{\bar{f}}^{\wp}} = u_{\bar{f}}$.

The above calculations show that

$$(\mathbb{F}^{\wp} * \mathbb{G}^{\wp})/(\mathbf{z})(\mathbb{F}^{\wp} * \mathbb{G}^{\wp}) \longrightarrow \mathbb{F} * \mathbb{G}, \quad \overline{g * e} \mapsto \bar{g} * \bar{e},$$

establishes an isomorphism of complexes, as desired.

In the final step of the proof we assume that I and J are arbitrary monomial ideals but $\mathbb{F}$ and $\mathbb{G}$ are not necessarily minimal. Then $\mathbb{F}$ can be written as a direct sum $\mathbb{F} = \mathbb{F}' \oplus \mathbb{H}$ of multigraded complexes with $\mathbb{F}'$ a minimal free resolution of S/I and $\mathbb{H}$ exact. Note that $\mathbb{H}$ is a direct sum of complexes of the form $\mathbb{D} : 0 \to Sg \to Sh \to 0$ whose differential maps g to h. Since $\mathbb{D} * \mathbb{G}$ is isomorphic to the mapping cone of the identity on $\mathbb{G}$ we see that $\mathbb{D} * \mathbb{G}$ is exact. It follows that $\mathbb{H} * \mathbb{G}$ is exact, and consequently $\mathbb{F} * \mathbb{G}$ has the same homology as $\mathbb{F}' * \mathbb{G}$. By the same argument we may replace $\mathbb{G}$ by a minimal multigraded free resolution of $\mathbb{G}'$ and thus obtain that

$$H_i(\mathbb{F} * \mathbb{G}) \cong H_i(\mathbb{F}' * \mathbb{G}') = \begin{cases} 0, & \text{if } \; i > 0, \\ S/(I + J), & \text{if } \; i = 0. \end{cases}$$

$\square$

3. Consequences

The product $\mathbb{F} * \mathbb{G}$ defined for multigraded free resolutions is associative, that is, we have

$$(\mathbb{F} * \mathbb{G}) * \mathbb{H} \cong \mathbb{F} * (\mathbb{G} * \mathbb{H})$$

for any three multigraded free resolutions. Thus if $I_j \subset S$ is a monomial ideal and $\mathbb{F}_j$ is a multigraded free S-resolution of S/I_j for $j = 1, \ldots, r$, then

$$\mathbb{F}_1 * \mathbb{F}_2 * \cdots * \mathbb{F}_r$$

is multigraded free S-resolution of $S/(I_1 + I_2 + \cdots + I_r)$.

Consider the following special case: let I be a monomial ideal with unique minimal monomial set of generators $G(I) = \{u_1, \cdots, u_r\}$, and set $I_j = (u_j)$ and $a_j = \deg u_j$ for $j = 1, \ldots, r$. Then

$$\mathbb{F}_j \colon 0 \longrightarrow S(-a_j) \xrightarrow{u_j} S \longrightarrow 0$$

is a multigraded free S-resolution of S/I_j, and so $\mathbb{T} = \mathbb{F}_1 * \mathbb{F}_2 * \cdots * \mathbb{F}_r$ is a multigraded free S-resolution of S/I. Indeed, $\mathbb{T}$ is the well-known Taylor resolution of S/I (cf. [4, Exercise 17.11]).

An obvious consequence of our construction is the following

Corollary 3.1. *Let I and J be monomial ideals in S. Then*

$$\beta_i(S/(I+J)) \leq \sum_{j=0}^{i} \beta_j(S/I)\beta_{i-j}(S/J).$$

For a graded ideal $L \subset S$ we set $M_i(L) = \max\{j\colon \operatorname{Tor}_i^S(K,L)_j \neq 0\}$. In other words, $M_i(L)$ is the highest shift in the ith step of the graded minimal free resolution of L. Furthermore we set $\operatorname{reg}_i(L) = M_i(L) - i$ for $i \geq 0$ and $\operatorname{reg}(L)_{-1} = 0$. The *regularity* of L is then defined to be $\max\{\operatorname{reg}_i(L)\colon i \geq 0\}$.

Corollary 3.1 implies the inequalities (a) described in the next corollaries. Inequality 3.2(b) concerning the regularity was conjectured by Terai [7] and proved in a special case.

Corollary 3.2 (Kalai, Meshulam)**.** *Let I and J be monomial ideals. Then*

(a) $\operatorname{proj\,dim}(I+J) \leq \operatorname{proj\,dim}(I) + \operatorname{proj\,dim}(J) + 1$;
(b) $\operatorname{reg}(I+J) \leq \operatorname{reg}(I) + \operatorname{reg}(J) - 1$.

Proof. It remains to prove statement (b). Since $\mathbb{F} * \mathbb{G}$ is a possibly non-minimal graded free resolution of $S/(I+J)$, we see that $M_i(I+J)$ is less than or equal to the maximal $\mathbb{Z}$-degree of a generator of $(\mathbb{F} * \mathbb{G})_{i+1}$.

Since $(\mathbb{F} * \mathbb{G})_{i+1} = \bigoplus_{j+k=i+1} F_j * G_k$ and since $\deg f * g \leq \deg f + \deg g$ for all homogeneous elements f and g, it follows that

$$\begin{aligned} \operatorname{reg}_i(I+J) &\leq \max_{j+k=i+1}\{M_{j-1}(I) + M_{k-1}(J)\} - i \\ &= \max_{j+k=i+1}\{\operatorname{reg}_{j-1}(I) + \operatorname{reg}_{k-1}(J)\} - 1. \end{aligned}$$

This implies the desired inequality. $\square$

From the exact sequence

$$0 \longrightarrow I \cap J \longrightarrow I \oplus J \longrightarrow I + J \longrightarrow 0$$

and Corollary 3.2 we deduce the following inequalities

Corollary 3.3 (Kalai, Meshulam)**.** *Let I and J be monomial ideals. Then*

(a) $\operatorname{proj\,dim}(I \cap J) \leq \operatorname{proj\,dim}(I) + \operatorname{proj\,dim}(J)$;
(b) $\operatorname{reg}(I \cap J) \leq \operatorname{reg}(I) + \operatorname{reg}(J)$.

In the case of monomial complete intersections, a proof of inequality (b) is given by Marc Chardin, Nguyen Cong Minh and Ngo Viet Trung [3].

For simplicial complexes the inequality of Corollary 3.1 has the following interpretation. Let Σ be a simplicial complex and U a subset of the vertex set. We denote by Σ_U the restriction of Σ to U, that is, the simplicial complex with faces $F \in \Sigma$ such that $F \subset U$. We fix a field and denote by $\tilde{H}_i(\Sigma)$ the ith reduced simplicial homology of Σ with respect to K, and by $\tilde{h}_i(\Sigma)$ the K-dimension of $\tilde{H}_i(\Sigma)$. With this notation we have

Corollary 3.4. *Let Δ and Γ be simplicial complexes on the vertex set $[n]$. Then*

$$\sum_W \tilde{h}_{|W|-i-1}((\Delta \cap \Gamma)_W) \leq \sum_{j+k=i} (\sum_U \tilde{h}_{|U|-j-1}(\Delta_U))(\sum_V \tilde{h}_{|V|-k-1}(\Gamma_V)).$$

Here the sums are taken over all subsets $U, V, W \subset [n]$

Proof. The inequality is a consequence of Corollary 3.1 and Hochster's formula

$$\dim_k \operatorname{Tor}_i^S(K; K[\Sigma]) = \sum_W \tilde{h}_{|W|-i-1}(\Sigma_W),$$

see [2, Theorem 5.5.1]. □

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